INTEGRAL VARIATIONAL PRINCIPLE OF MECHANICS

(INTEGRAL'NYI VARIATSIONNYI PRINTSIP MEKHANIKI)

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Many works have been devoted to the variational principles of mechanics; of these we mention only the monographs [1 to 4].

The well known integral variational formulations of problems of mechanics assume that the position of the mechanical system is specified at the end of the time interval under consideration. However, the final position is usually not known, while the initial position and velocity, are. Variational formulations of this problem with given initial conditions are presented here for linear systems with equations having constant coefficients.

1. The first form of the variational principle. Let us consider a system of particles which has n degrees of freedom. We shall denote the generalized coordinates by q_1, $q_2, \ldots, q_n$. It will be assumed that the kinetic energy T and the potential energy of the system proper U are representable as positive definite quadratic forms with constant coefficients in the generalized velocities and the generalized coordinates, respectively

$$T = \frac{1}{2} \sum_{i,\,k=1}^{n} a_{ik} q_i^{\cdot} q_k^{\cdot}, \qquad U = \frac{1}{2} \sum_{i,\,k=1}^{n} c_{ik} q_i q_k \tag{1.1}$$

The external forces $f_1, \ldots, f_n$ act on the system. The potential of these forces is

$$V = -\sum_{i=1}^{n} f_i q_i \tag{1.2}$$

The Lagrangian then has the form

$$L = \frac{1}{2} \sum_{i,\,k=1}^{n} a_{ik} q_i^{\cdot} q_k^{\cdot} - \frac{1}{2} \sum_{i,\,k=1}^{n} c_{ik} q_i q_k + \sum_{i=1}^{n} f_i q_i \tag{1.3}$$

If the positions of the system at the times $t = 0$ and $t = \tau$ are known, then, according to Hamilton's principle, the system moves between these positions in such a way that

$$\delta \int_0^{\tau} L(q, q^{\cdot}, t)\, dt = 0 \tag{1.4}$$

For given positions at the times $t = 0$ and $t = \tau$, it follows from the condition (1.4) that the equations of motion of the system are

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} = 0 \quad \text{or} \quad \sum_{k=1}^{n} (a_{ik} q_k^{\cdot\cdot} + c_{ik} q_k) = f_i \qquad (i = 1, 2, \ldots, n). \tag{1.5}$$

Let us now turn to the case in which the position and velocity of the system are specified at the initial time $t = 0$. Without loss of generality we may take

$$q_i(0) = 0, \qquad q_i^{\cdot}(0) = 0 \tag{1.6}$$

Taking into account the form of the Lagrangian (1.3), we shall examine the functional

$$A = \int_0^{\tau} K^*(q, q^*, q^{\cdot}, q^{\cdot *}, t)\, dt$$

$$K^* = \sum_{i,\,k=1}^{n} a_{ik} q_i^{\cdot} q_k^{\cdot *} - \sum_{i,\,k=1}^{n} c_{ik} q_i q_k^* + \sum_{i=1}^{n} f_i^* q_i + \sum_{i=1}^{n} f_i q_i^* \tag{1.7}$$

We assume here that the functions q_i and q_i^* are mutually independent functional arguments and that the f_i^* as well as the f_i are given functions. The particular form of the functions f_i^* will be determined later.

We consider the first variation of the functional (1.7)

$$\delta A = \int_0^{\tau} \sum_{i=1}^{n} \left\{ \left(\frac{\partial K^*}{\partial q_i^*} - \frac{d}{dt}\frac{\partial K^*}{\partial q_i^{\cdot *}} \right) \delta q_i^* + \left(\frac{\partial K^*}{\partial q_i} - \frac{d}{dt}\frac{\partial K^*}{\partial q_i^{\cdot}} \right) \delta q_i \right\} dt + \\ + \sum_{i=1}^{n} \left(\frac{\partial K^*}{\partial q_i^{\cdot *}} \delta q_i^* + \frac{\partial K^*}{\partial q_i^{\cdot}} \delta q_i \right) \Bigg|_{t=0}^{t=\tau} \tag{1.8}$$

The following notation is introduced:

$$L^* = \frac{1}{2} \sum_{i,\,k=1}^{n} a_{ik} q_i^{\cdot *} q_k^{\cdot *} - \frac{1}{2} \sum_{i,\,k=1}^{n} c_{ik} q_i^* q_k^* + \sum_{i=1}^{n} f_i^* q_i^*$$

Then

$$\frac{\partial K^*}{\partial q_i^{\cdot *}} = \frac{\partial L}{\partial q_i^{\cdot}}, \qquad \frac{\partial K^*}{\partial q_i^{\cdot}} = \frac{\partial L^*}{\partial q_i^{\cdot *}}, \qquad \frac{\partial K^*}{\partial q_i^*} = \frac{\partial L}{\partial q_i}, \qquad \frac{\partial K^*}{\partial q_i} = \frac{\partial L^*}{\partial q_i^*} \tag{1.9}$$

or

$$\frac{\partial K^*}{\partial q_i^{\cdot *}} = \sum_{k=1}^{n} a_{ik} q_k^{\cdot} \qquad \frac{\partial K^*}{\partial q_i^{\cdot}} = \sum_{k=1}^{n} a_{ik} q_k^{\cdot *}$$

After making use of Equations (1.9), the variation (1.8) can be represented in the form

$$\delta A = \int_0^{\tau} \sum_{i=1}^{n} \left\{ \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} \right) \delta q_i^* + \left(\frac{\partial L^*}{\partial q_i^*} - \frac{d}{dt}\frac{\partial L^*}{\partial q_i^{\cdot *}} \right) \delta q_i \right\} dt + \\ + \sum_{i,\,k=1}^{n} a_{ik} \left[q_i^{\cdot}(\tau)\, \delta q_k^*(\tau) + q_i^{\cdot *}(\tau)\, \delta q_k(\tau) - q_i^{\cdot}(0)\, \delta q_k^*(0) - q_i^{\cdot *}(0)\, \delta q_k(0) \right] \tag{1.10}$$

It follows from Equation (1.10) that $\delta A = 0$ if the functions q_i satisfy the equations (1.5) and the initial conditions (1.6) and the functions q_i^* satisfy the equations

$$\frac{d}{dt}\frac{\partial L^*}{\partial q_i^{\cdot *}} - \frac{\partial L^*}{\partial q_i^*} = 0 \quad \text{or} \quad \sum_{k=1}^{n} (a_{ik} q_k^{\cdot\cdot *} + c_{ik} q_k^*) = f_i^* \tag{1.11}$$

and the conditions

$$q_i^* (\tau) = 0, \qquad q_i^{\cdot *} (\tau) = 0 \tag{1.12}$$

We now transform the auxiliary problem (1.11) and (1.12) to the form of the problem under study (1.5) and (1.6). We introduce the new variable

$$\eta = \tau - t \tag{1.13}$$

and define

$$r_i(\eta) = q_i^*(\tau - \eta), \qquad g_i(\eta) = f_i^* (\tau - \eta)$$
$$\Lambda (r, r', \eta) = L^* [q^* (\tau - \eta), \quad q^{\cdot *} (\tau - \eta), \tau - \eta] \tag{1.14}$$

We remark that

$$r_i' = \frac{dr_i}{d\eta} = - q_i^{\cdot *} (\tau - \eta) \tag{1.15}$$

According to the relation (1.9), we have

$$\Lambda (r, r', \eta) = \frac{1}{2} \sum_{i,k=1}^{n} a_{ik} r_i' r_k' - \frac{1}{2} \sum_{i,k=1}^{n} c_{ik} r_i r_k + \sum_{i=1}^{n} g_i r_i \tag{1.16}$$

Equations (1.11) and the conditions (1.12) now take the form

$$\frac{d}{\partial\eta}\frac{\partial \Lambda}{\partial r_i'} - \frac{\partial \Lambda}{\partial r_i} = 0 \quad \text{or} \quad \sum_{k=1}^{n} (a_{ik} r_k'' + c_{ik} r_k) = g_i \tag{1.17}$$

$$r_i(0) = 0, \qquad r_i' (0) = 0 \tag{1.18}$$

in the new notation.

The equations (1.17) and the conditions (1.18) which have been obtained differ from (1.5) and the conditions (1.6) only as regards the notation for the variables. If we now denote

$$\eta = t, \quad r_i = q_i, \quad g_i = f_i \tag{1.19}$$

the equations of the auxiliary problem (1.17) with the initial conditions (1.18) coincide completely with the equations of the problem under study (1.5) and initial conditions (1.6).

The relations (1.14) and (1.19) permit us to express q_i^* and f_i^* in terms of the functions q_i and f_i

$$q_i^* (\tau - t) = q_i(t), \qquad f_i^* (\tau - t) = f_i(t) \tag{1.20}$$

It is easy to verify that the equations

$$q_i^* (t) = q_i(\tau - t), \qquad f_i^* (t) = f_i(\tau - t) \tag{1.21}$$

follow from (1.20).

Substituting the obtained functions q_i^* and f_i^* into the functional (1.7), we obtain the following variational principle of mechanics. For given initial position and velocity, the true motion of the system in the time interval $(0, T)$ is such that the integral

$$A = \int_0^\tau K\,[q(t),\ q(\tau - t),\ q^{\cdot}(t),\ q^{\cdot}(\tau - t),\ t]\,dt \tag{1.22}$$

is stationary. Here the function K has the form

$$K = -\sum_{i,\,k=1}^{n} a_{ik} q_i^{\cdot}(t)\, q_k^{\cdot}(\tau - t) - \sum_{i,\,k=1}^{n} c_{ik} q_i(t)\, q_k(\tau - t) + 2\sum_{i=1}^{n} f_i(t)\, q_i(\tau - t) \tag{1.23}$$

2. The second form of the variational principle. We shall formulate a variational principle of mechanics for the case in which the motion of the system is described with the aid of generalized coordinates and momenta. The problem under consideration is defined by Hamilton's equations

$$q_i^{\cdot} = \frac{\partial H}{\partial p_i}, \qquad p_i^{\cdot} = -\frac{\partial H}{\partial q_i} \tag{2.1}$$

$$H(q,\,p,\,t) = \frac{1}{2}\sum_{i,\,k=1}^{n} a_{ik}^{(-1)} p_i p_k + \frac{1}{2}\sum_{i,\,k=1}^{n} c_{ik} q_i q_k - \sum_{i=1}^{n} f_i q_i \tag{2.2}$$

with the initial conditions

$$q_i(0) = 0, \qquad p_i(0) = 0 \tag{2.3}$$

The validity of the following variational principle can be proved for the present problem. For given initial positions and momenta the true motion of the system in the time internal $(0,\ T)$ is such that the integral

$$J = \int_0^\tau \Big\{ \sum_{i=1}^{n} p_i(\tau - t)\, q_i^{\cdot}(t) + G\,[q(t),\ q(\tau - t),\ p(t),\ p(\tau - t),\ t] \Big\}\, dt \tag{2.4}$$

is stationary. Here the function G has the form

$$G = -\frac{1}{2}\sum_{i,\,k=1}^{n} a_{ik}^{(-1)} p_i(t)\, p_k(\tau - t) + \frac{1}{2}\sum_{i,\,k=1}^{n} c_{ik} q_i(t)\, q_k(\tau - t) - \sum_{i=1}^{n} f_i(\tau - t)\, q_i(t) \tag{2.5}$$

To prove this principle we take the first variation of the functional (2.4)

$$\begin{aligned}\delta J = -\int_0^\tau \Big\{ \sum_{i=1}^{n} \Big[p_i^{\cdot}(\tau - t) - \frac{\partial G}{\partial q_i} \Big] \delta q_i(t) \,+ \\ + \sum_{i=1}^{n} \Big[q_i^{\cdot}(\tau - t) - \frac{\partial G}{\partial p_i} \Big] \delta p_i(t) \Big\}\, dt - \sum_{i=1}^{n} p_i(\tau - t)\ \delta q_i(t)\,\Big|_{t=0}^{t=\tau}\end{aligned} \tag{2.6}$$

Taking account of the equation

$$\int_0^\tau u(t)\, v(\tau - t)\, dt = \int_0^\tau u(\tau - t)\, v(t)\, dt \tag{2.7}$$

we obtain from (2.2) and (2.5)

$$\frac{\partial G}{\partial q_i} = \frac{\partial H\,[q(\tau - t),\ p(\tau - t),\ \tau - t]}{\partial q_i(\tau - t)}, \qquad \frac{\partial G}{\partial p_i} = -\frac{\partial H\,[q(\tau - t),\ p(\tau - t),\ \tau - t]}{\partial p_i(\tau - t)} \tag{2.8}$$

By virtue of the relations (2.8), the variation (2.6) is equal to zero if the conditions (2.3) and the equations

$$\begin{aligned} q_i^{\cdot}(\tau - t) + \frac{\partial H\,[q(\tau - t),\ p(\tau - t),\ \tau - t]}{\partial p_i(\tau - t)} &= 0 \\ p_i^{\cdot}(\tau - t) - \frac{\partial H\,[q(\tau - t),\ p(\tau - t),\ \tau - t]}{\partial q_i(\tau - t)} &= 0 \end{aligned} \tag{2.9}$$

are satisfied. Since these equations differ from the Hamilton's equations (2.1) only as regards the notation of the independent variable, the variational principle is proved.

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